On stability of line solitons for the KP-II equation (Mathematical Analysis in Fluid and Gas Dynamics)

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1 Introduction

The KP-II equation

$$\partial_x(\partial_t u+\partial_x^3 u+3\partial_x(u^2))+3\partial_y^2 u=0 \quad \text{for } t>0 \text{ and } (x,y)\in\mathbb{R}^2, \tag{1.1}$$

is a generalization to two spatial dimensions of the KdV equation

$$\partial_t u+\partial_x^3 u+3\partial_x(u^2)=0\,, \tag{1.2}$$

and has been derived as a model in the study of the transverse stability of solitary wave solutions to the KdV equation with respect to two dimensional perturbation when the surface tension is weak or absent (see [14]).

The global well-posedness of (1.1) in $H^s(\mathbb{R}^2)$ $(s\geq 0)$ on the background of line solitons has been studied by Molinet, Saut and Tzvetkov [29] whose proof is based on the work of Bourgain [5]. For the other contributions on the Cauchy problem of the KP-II equation, see e.g. [10, 11, 12, 13, 34, 35, 36, 37] and the references therein.

Let

$$\varphi_c(x)\equiv c\cosh^{-2}\left(\sqrt{\frac{c}{2}}\,x\right),\quad c>0.$$

Then φ_c is a solution of

$$\varphi_c''-2c\varphi_c+3\varphi_c^2=0\,, \tag{1.3}$$

and $\varphi_c(x-2ct)$ is a solitary wave solution of the KdV equation (1.2) and a line soliton solution of (1.1) as well.

Let us briefly explain known results on stability of 1-solitons for the KdV equation first. Stability of the 1-soliton $\varphi_c(x-2ct)$ of (1.2) was proved by [2, 4, 39] using the fact that φ_c is a minimizer of the Hamiltonian on the manifold $\{u\in H^1(\mathbb{R})\mid \|u\|_{L^2(\mathbb{R})}=\|\varphi_c\|_{L^2(\mathbb{R})}\}$.

Solitary waves of the KdV equation travel at speeds faster than the maximum group velocity of linear waves and the larger solitary wave moves faster to the right. Using this property, Pego and Weinstein [31] prove asymptotic stability of solitary wave solutions of (1.2) in an exponentially weighted space. Later, Martel and Merle established the Liouville theorem for the generalized KdV equations by using a virial type identity and prove the asymptotic stability of solitary waves in $H^1_{loc}(\mathbb{R})$ (see e.g. [18]). For stability of multi-solitons of the generalized KdV equations, see [19].

For the KP-II equation, its Hamiltonian is infinitely indefinite and the variational approach such as [9] is not applicable. Hence it seems natural to study stability of line solitons using strong linear stability of line solitons as in [31]. Spectral transverse stability of line solitons of (1.1) has been studied by [1, 6]. Alexander *et al.* [1] proved that the spectrum of the linearized operator in $L^2(\mathbb{R}^2)$ consists of the entire imaginary axis. On the other hand, in an exponentially weighted space where the size of perturbations are biased in the direction of motion, the spectrum of the linearized operator consists of a curve of resonant continuous eigenvalues which goes through 0 and the set of continuous spectrum which locates in the stable half plane and is away from the imaginary axis (see [6, 22]). The former one appears because line solitons are not localized in the transversal direction and 0, which is related to the symmetry of line solitons, cannot be an isolated eigenvalue of the linearized operator. Such a situation is common with planer traveling wave solutions for the heat equation. See e.g. [15, 17, 40].

Using the inverse scattering method, Villarroel and Ablowitz [38] studied solutions of around line solitons for (1.1). Recently, Mizumachi [22] has proved transversal stability of line soliton solutions of (1.1) for exponentially localized perturbations. The idea is to use the exponential decay property of the linearized equation satisfying a secular term condition and describe variations of local amplitudes and local inclinations of the crest of modulating line solitons by a system of Burgers equations.

Now let us introduce our results.

Theorem 1.1. *Let $c_0 > 0$ and $u(t,x,y)$ be a solution of* (1.1) *satisfying $u(0,x,y) = \varphi_{c_0}(x) + v_0(x,y)$. There exist positive constants ε_0 and C satisfying the following: if $v_0 \in H^{1/2}(\mathbb{R}^2) \cap \partial_x L^2(\mathbb{R}^2)$ and $\|v_0\|_{L^2(\mathbb{R}^2)} + \||D_x|^{1/2}v_0\|_{L^2} + \||D_x|^{-1/2}|D_y|^{1/2}v_0\|_{L^2(\mathbb{R}^2)} < \varepsilon_0$ then there exist C^1-functions $c(t,y)$ and $x(t,y)$ such that for every $t \geq 0$ and $k \geq 0$,*

$$\|u(t,x,y) - \varphi_{c(t,y)}(x - x(t,y))\|_{L^2(\mathbb{R}^2)} \leq C\|v_0\|_{L^2}\,, \tag{1.4}$$

$$\|c(t,\cdot) - c_0\|_{H^k(\mathbb{R})} + \|\partial_y x(t,\cdot)\|_{H^k(\mathbb{R})} + \|x_t(t,\cdot) - 2c(t,\cdot)\|_{H^k(\mathbb{R})} \leq C\|v_0\|_{L^2}\,, \tag{1.5}$$

$$\lim_{t\to\infty} \left(\|\partial_y c(t,\cdot)\|_{H^k(\mathbb{R})} + \left\|\partial_y^2 x(t,\cdot)\right\|_{H^k(\mathbb{R})}\right) = 0\,, \tag{1.6}$$

and for any$R > 0$,

$$\lim_{t\to\infty} \left\|u(t, x + x(t,y), y) - \varphi_{c(t,y)}(x)\right\|_{L^2((x>-R)\times\mathbb{R}_y)} = 0\,. \tag{1.7}$$

Theorem 1.2. *Let $c_0 > 0$ and $s > 1$. Suppose that u is a solutions of* (1.1) *satisfying $u(0,x,y) = \varphi_{c_0}(x) + v_0(x,y)$. Then there exist positive constants ε_0 and C such that if $\|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)} < \varepsilon_0$, there exist $c(t,y)$ and $x(t,y)$ satisfying* (1.6), (1.7) *and*

$$\|u(t,x,y) - \varphi_{c(t,y)}(x - x(t,y))\|_{L^2(\mathbb{R}^2)} \leq C\|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)}\,, \tag{1.8}$$

$$\|c(t,\cdot) - c_0\|_{H^k(\mathbb{R})} + \|\partial_y x(t,\cdot)\|_{H^k(\mathbb{R})} + \|x_t(t,\cdot) - 2c(t,\cdot)\|_{H^k(\mathbb{R})} \leq C\|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)} \tag{1.9}$$

for every $t \geq 0$ and $k \geq 0$.

Remark 1.1. By (1.5) and (1.6),

$$\lim_{t\to\infty} \sup_{y\in\mathbb{R}}(|c(t,y) - c_0| + |x_y(t,y)|) = 0\,,$$

and as $t \to \infty$, the modulating line soliton $\varphi_{c(t,y)}(x - x(t,y))$ converges to a y-independent modulating line soliton $\varphi_{c_0}(x - x(t,0))$ in $L^2(\mathbb{R}_x \times (|y| \leq R))$ for any $R > 0$. Hence it follows from (1.7) that

$$\lim_{t\to\infty} \|u(t, x + x(t,0), y) - \varphi_{c_0}(x)\|_{L^2((x>-R)\times(|y|\leq R))} = 0\,.$$

We remark that the phase shift $x(t,y)$ in (1.4) and (1.7) cannot be uniform in y because of the variation of the local phase shift around $y = \pm 2\sqrt{2c_0}t + O(\sqrt{t})$. See Theorems 1.4 and 1.5 in [22].

Remark 1.2. The KP-II equation has no localized solitary waves (see [7, 8]). On the other hand, the KP-I equation has stable ground states (see [8, 16]) and line solitons of the KP-I equation are unstable (see [32, 33, 41]).

Now let us explain our strategy of the proof. To prove stability of line solitons in [22], we rely on the fact that solutions of the linearized equation decay exponentially in exponentially weighted norm as $t \to \infty$ if data are orthogonal to the adjoint resonant continuous eigenmodes. To describe the behavior of solutions around a line soliton, we represent them by using an ansatz

$$u(t,x,y) = \varphi_{c(t,y)}(z) - \psi_{c(t,y)}(z+3t) + v(t,z,y)\,, \quad z = x - x(t,y)\,, \tag{1.10}$$

where $c(t,y)$ and $x(t,y)$ are the local amplitude and the local phase shift of the modulating line soliton $\varphi_{c(t,y)}(x - x(t,y))$ at time t along the line parallel to the x-axis and $\psi_{c(t,y)}$ is an auxiliary function so that

$$\int_{\mathbb{R}} v(t,z,y)\,dz = \int_{\mathbb{R}} v(0,z,y)\,dz \quad \text{for any } y \in \mathbb{R}.$$

One of the key step is to prove $\|v(t)\|_{L^2_{loc}}$ is square integrable in time. In [22], we impose a non secular condition on $v(t)$ such that the perturbation $v(t)$ is orthogonal to the adjoint resonant eigenfunctions in order to apply the strong linear stability property of line solitons (see Proposition 2.1 in Section 2)to v. Since the adjoint resonant eigenfunctions grow exponentially as $x \to \infty$, the secular term condition is not feasible for $v(t)$ which is not exponentially localized as $x \to \infty$. Following the idea of [21, 24, 25], we split the perturbation $v(t)$ into a sum of a small solution $v_1(t)$ of (1.1) satisfying $v_1(0) = v_0$ and the remainder part $v_2(t)$. As is the same with other long wave models in [21, 24], the solitary wave part moves faster than the freely propagating freely propagating perturbations and the localized L^2-norms of v_1 are square integrable in time thanks to the virial identity (see [7]).

The remainder part $v_2(t)$ is exponentially localized as $x \to \infty$ and is mainly driven by the interaction between v_1 and the line soliton. We impose the secular term condition on v_2 to apply the linear stability estimate. To estimate $\|e^{\alpha z}v_2(t)\|_{L^2}$ with small $\alpha > 0$, we use the semigroup estimate introduced in Section 2 to estimate the low frequencies in y and apply a virial type estimate to estimate high frequencies in y to avoid a loss of derivatives caused by the use of the semigroup estimate.

Since we split the perturbation v into two parts v_1 and v_2, in the the virial identity of v_2, we cannot cancel the derivative of the nonlinear term $\partial_x(v_1v_2)$ by integration by parts and we need a time global bound of $\|v_1(t)\|_{L^3}$. For the purpose , we use the nonlinear scattering theory in [12] which gives a time global bound for L^p-norms with $p > 2$ if $v_1(0) = v_0 \in |D_x|^{1/2}L^2(\mathbb{R}^2)$ is small and v_0 is sufficiently smooth.

Another point is that we do not assume v_0 is integrable in y. For this reason, the modulation parameters $\tilde{c}(t,y) := c(t,y) - c_0$ and $x_y(t,y)$ are not necessarily pseudo-measures and we are not able to estimate $\mathcal{F}^{-1}L^\infty - L^2$ estimates for $\tilde{c}$ and x_y. Instead, we use the monotonicity formula to obtain time global bounds for $\tilde{c}(t)$ and $x_y(t)$. Since the terms related to $v_1(t)$ are merely square integrable in time and cubic terms that appear in the energy identity are not necessarily integrable in time, we use a change of variables to eliminate these terms to obtain time global estimates.

In this report, we explain the strategy of the proof of Theorem 1.1 and .

Finally, let us introduce several notations. For Banach spaces V and W, let $B(V,W)$ be the space of all linear continuous operators from V to W and let $\|T\|_{B(V,W)} = \sup_{\|x\|_V=1}\|Tu\|_W$ for $T \in B(V,W)$. We abbreviate $B(V,V)$ as $B(V)$. For $f \in \mathcal{S}(\mathbb{R}^n)$ and $m \in \mathcal{S}'(\mathbb{R}^n)$, let

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x)e^{-ix\xi}\,dx\,,$$
$$(\mathcal{F}^{-1}f)(x) = \check{f}(x) = \hat{f}(-x)\,, \quad (m(D_x)f)(x) = (2\pi)^{-n/2}(\check{m} * f)(x)\,.$$

We use $a \lesssim b$ and $a = O(b)$ to mean that there exists a positive constant such that $a \le Cb$. Various constants will be simply denoted by C and C_i ($i \in \mathbb{N}$) in the course of the calculations. We denote $\langle x\rangle = \sqrt{1+x^2}$ for $x \in \mathbb{R}$.

2 Linear stability of line solitons

To prove stability of line solitons for the KP-II equation, we use linear stability property of the line-solitons as Pego and Weinstein [31] did for the KdV equation.. We recollect decay estimates of the semigroup generated by the linearized operator around a 1-line soliton and some function spaces to analyze modulation parameters of line solitons.

Since (1.1) is invariant under the scaling $u \mapsto \lambda^2u(\lambda^3t, \lambda x, \lambda^2 y)$, we may assume $c_0 = 2$ in Theorems 1.1 and 1.2 without loss of generality. Let

$$\varphi = \varphi_2\,, \quad \mathcal{L} = -\partial_x^3 + 4\partial_x - 3\partial_x^{-1}\partial_y^2 - 6\partial_x(\varphi\cdot)\,.$$

We remark that $e^{t\mathcal{L}}$ is a C^0-semigroup on $X := L^2(\mathbb{R}^2; e^{2\alpha x}dxdy)$ for any $\alpha > 0$ because $\mathcal{L}_0 := -\partial_x^3 + 4\partial_x - 3\partial_x^{-1}\partial_y^2$ is m-dissipative on X and $\mathcal{L} - \mathcal{L}_0$ is infinitesimally small with respect to $\mathcal{L}_0$.

Using Plancherel's theorem, we have $\|f\|_X = \|\hat{f}(\cdot + i\alpha, \cdot)\|_{L^2(\mathbb{R}^2)}$ and

$$\|e^{t\mathcal{L}_0}f\|_X \le e^{-\alpha(4-\alpha^2)t}\|f\|_X\,. \tag{2.1}$$

Solutions of $\partial_t u = \mathcal{L}u$ satisfying a *secular term condition* decay like solutions to the free equation $\partial_t u = \mathcal{L}_0 u$. To be more precise, let us introduce a family of continuous resonant eigenvalues near 0 and the corresponding continuous eigenfunctions of the linearized operator $\mathcal{L}$. Let

$$\beta(\eta) = \sqrt{1+i\eta}\,, \quad \lambda(\eta) = 4i\eta\beta(\eta)\,,$$
$$g(x,\eta) = \frac{-i}{2\eta\beta(\eta)}\partial_x^2(e^{-\beta(\eta)x}\operatorname{sech}x), \quad g^*(x,\eta) = \partial_x(e^{\beta(-\eta)x}\operatorname{sech}x)\,.$$

Then

$$\mathcal{L}(\eta)g(x,\pm\eta) = \lambda(\pm\eta)g(x,\pm\eta)\,, \quad \mathcal{L}(\eta)^*g^*(x,\pm\eta) = \lambda(\mp\eta)g^*(x,\pm\eta)\,.$$

Now we define a spectral projection to the resonant eigenmodes $\{g_\pm(x,\eta)\}$. Let

$$g_1(x,\eta) = 2\Re g(x,\eta)\,, \quad g_2(x,\eta) = -2\eta\Im g(x,\eta)\,,$$
$$g_1^*(x,\eta) = \Re g^*(x,\eta)\,, \quad g_2^*(x,\eta) = -\eta^{-1}\Im g^*(x,\eta)\,,$$

and $P_0(\eta_0)$ be a projection to resonant modes defined by

$$P_0(\eta_0)f(x,y)=\frac{1}{2\pi}\sum_{k=1,2}\int_{-\eta_0}^{\eta_0} a_k(\eta)g_k(x,\eta)e^{iy\eta}\,d\eta\,,$$

$$\begin{aligned}a_k(\eta)=&\int_{\mathbb{R}}\lim_{M\to\infty}\left(\int_{-M}^{M} f(x_1,y_1)e^{-iy_1\eta}\,dy_1\right)\overline{g_k^*(x_1,\eta)}\,dx_1\\ =&\sqrt{2\pi}\int_{\mathbb{R}}(\mathcal{F}_y f)(x,\eta)\overline{g_k^*(x,\eta)}\,dx\,.\end{aligned}$$

For η_0 and M satisfying $0<\eta_0\le M\le\infty$, let

$$\begin{aligned}P_1(\eta_0,M)u(x,y):=&\frac{1}{2\pi}\int_{\eta_0\le|\eta|\le M}\int_{\mathbb{R}} u(x,y_1)e^{i\eta(y-y_1)}dy_1d\eta\,,\\ P_2(\eta_0,M):=&P_1(0,M)-P_0(\eta_0)\,.\end{aligned}$$

Then we have the following.

Proposition 2.1. ([22, Proposition 3.2 and Corollary 3.3]) *Let $\alpha\in(0,2)$ and η_1 be a positive number satisfying $\Re\beta(\eta_1)-1<\alpha$. Then there exist positive constants K and b such that for any $\eta_0\in(0,\eta_1]$, $M\ge\eta_0$, $f\in X$ and $t\ge 0$,*

$$\|e^{t\mathcal{L}}P_2(\eta_0,M)f\|_X\le Ke^{-bt}\|f\|_X\,.$$

Moreover, there exist positive constants K' and b' such that for $t>0$,

$$\begin{aligned}\|e^{t\mathcal{L}}P_2(\eta_0,M)\partial_x f\|_X\le K'e^{-b't}t^{-1/2}\|e^{ax}f\|_X\,,\\ \|e^{t\mathcal{L}}P_2(\eta_0,M)\partial_x f\|_X\le K'e^{-b't}t^{-3/4}\|e^{ax}f\|_{L^1_xL^2_y}\,.\end{aligned}$$

3 Decomposition of the perturbed line soliton

Let us decompose a solution around a line soliton solution $\varphi(x-4t)$ into a sum of a modulating line soliton and a non-resonant dispersive part plus a small wave which is caused by amplitude changes of the line soliton:

$$u(t,x,y)=\varphi_{c(t,y)}(z)-\psi_{c(t,y),L}(z+3t)+v(t,z,y)\,,\quad z=x-x(t,y)\,,\tag{3.1}$$

where $\psi_{c,L}(x)=2(\sqrt{2c}-2)\psi(x+L)$, $\psi(x)$ is a nonnegative function such that $\psi(x)=0$ if $|x|\ge 1$ and that $\int_{\mathbb{R}}\psi(x)\,dx=1$ and $L>0$ is a large constant to be fixed later. The modulation parameters $c(t_0,y_0)$ and $x(t_0,y_0)$ denote the maximum height and the phase shift of the modulating line soliton $\varphi_{c(t,y)}(x-x(t,y))$ along the line $y=y_0$ at the time $t=t_0$, and $\psi_{c,L}$ is an auxiliary function such that

$$\int_{\mathbb{R}}\psi_{c,L}(x)\,dx=\int_{\mathbb{R}}(\varphi_c(x)-\varphi(x))\,dx\,.\tag{3.2}$$

Since a localized solution to KP-type equations satisfies $\int_{\mathbb{R}}u(t,x,y)\,dx=0$ for any $y\in\mathbb{R}$ and $t>0$ (see [27]), it is natural to expect small perturbations appear in the rear of the solitary wave if the solitary wave is amplified.

To utilize exponential linear stability of line solitons for solutions that are not exponentially localized in space, we further decompose v into a small solution of (1.1) and an exponentially localized part following the idea of [21] (see also [24, 26]). Let $\tilde{v}_1$ be a solution of

$$\begin{cases} \partial_t \tilde{v}_1 + \partial_x^3 \tilde{v}_1 + 3\partial_x(\tilde{v}_1^2) + 3\partial_x^{-1}\partial_y^2 \tilde{v}_1 = 0\,, \\ \tilde{v}_1(0,x,y) = v_0(x,y)\,, \end{cases} \tag{3.3}$$

and

$$v_1(t,z,y) = \tilde{v}_1(t, z + x(t,y), y)\,, \quad v_2(t,z,y) = v(t,z,y) - v_1(t,z,y)\,. \tag{3.4}$$

Obviously, we have $v_2(0) = 0$ and $v_2(t) \in X := L^2(\mathbb{R}^2; e^{2\alpha z} dz dy)$ for $t \geq 0$ as long as the decomposition (3.1) persists. Indeed, we have the following.

Lemma 3.1. *Let $v_0 \in H^{1/2}(\mathbb{R}^2)$ and $\tilde{v}_1(t)$ be a solution of* (3.3). *Suppose $u(t)$ is a solution of* (1.1) *satisfying $u(0,x,y) = \varphi(x) + v_0(x,y)$. Then for any $\alpha \in [0,1)$, $w(t,x,y) = u(t, x+4t, y) - \varphi(x) - \tilde{v}_1(t, x+4t, y)$ satisfies*

$$w \in C([0,\infty); X)\,, \tag{3.5}$$

$$\partial_x w\,, \ \partial_x^{-1}\partial_y w \in L^2(0,T;X) \quad \textit{for any } T > 0. \tag{3.6}$$

Moreover, the mapping

$$H^{1/2}(\mathbb{R}^2) \ni v_0 \mapsto w \in C([0,T]; X)$$

is continuous.

Note that by [29], we know $\partial_x w$, $\partial_x^{-1}\partial_y w \in L_x^\infty L_{ty}^2$ in advance.

To fix the decomposition (3.1), we impose that $v_2(t,z,y)$ is symplectically orthogonal to low frequency resonant modes. More precisely, we impose the constraint that for $k = 1$, 2,

$$\lim_{M\to\infty} \int_{-M}^{M} \int_{\mathbb{R}} v_2(t,z,y) \overline{g_k^*(z,\eta,c(t,y))} e^{-iy\eta}\, dz dy = 0 \quad \text{in } L^2(-\eta_0, \eta_0), \tag{3.7}$$

where $g_1^*(z,\eta,c) = c g_1^*(\sqrt{c/2}z, \eta)$ and $g_2^*(z,\eta,c) = \frac{c}{2} g_2^*(\sqrt{c/2}z, \eta)$.

Since $w(0) = 0$ and $w \in C([0,\infty); X)$, a pair of (v_2, c, x) satisfying (3.1), (3.4) and (3.7) exists at least locally in time.

Let Y and Z be closed subspaces of $L^2(\mathbb{R})$ such that for an $\eta_0 > 0$,

$$Y = \mathcal{F}_\eta^{-1} Z\,, \quad Z = \{f \in L^2(\mathbb{R}) \mid \operatorname{supp} f \subset [-\eta_0, \eta_0]\}\,.$$

Using the implicit function theorem, we have the following.

Proposition 3.2. *Let $\alpha \in (0,1)$ and let δ_0 and L is a large positive number. Then there exists $T > 0$ and $v_2(t,z,y)$, $\tilde{c}(t,y) := c(t,y) - 2$ and $\tilde{x}(t,y) := x(t,y) - 4t$ such that*

$$(v_2, \tilde{c}, \tilde{x}) \in C([0,T]; X \times Y \times Y)\,,$$

and satisfy (3.1), (3.4) *and* (3.7) *for $t \in [0,T)$. Moreover, there exist positive constants δ_1 and δ_2 such that if*

$$\sup_{t\in[0,T)} \|v_2(t)\|_X \leq \delta_1\,, \quad \sup_{t\in[0,T)} \|\tilde{c}(t)\|_Y < \delta_2\,, \quad \sup_{t\in[0,T)} \|\tilde{x}(t)\|_Y < \infty\,, \tag{3.8}$$

then either $T = \infty$ or T is not the maximal time of the decomposition (3.1) *satisfying* (3.4), (3.7) *and* (3.8).

4 Modulation equations

Next, we will derive a system of PDEs which describe the motion of modulation parameters $c(t,y)$ and $x(t,y)$. Substituting $\tilde{v}_1(t,x,y)=v_1(t,z,y)$ with $z=x-x(t,y)$ into (1.1), we have

$$\partial_t v_1 - 2c\partial_z v_1 + \partial_z^3 v_1 + 3\partial_z^{-1}\partial_y^2 v_1 = \partial_z(N_{1,1}+N_{1,2}) + N_{1,3}\,, \tag{4.1}$$

where $N_{1,1}=-3v_1^2$, $N_{1,2}=\{x_t-2c-3(x_y)^2\}v_1$ and $N_{1,3}=6\partial_y(x_yv_1)-3x_{yy}v_1$. Substituting the ansatz (3.1) into (1.1), we have

$$\partial_t v = \mathcal{L}_c v + \ell + \partial_z(N_1+N_2) + N_3\,, \tag{4.2}$$

where $\mathcal{L}_c v = -\partial_z(\partial_z^2 - 2c + 6\varphi_c)v - 3\partial_z^{-1}\partial_y^2$, $\ell=\ell_1+\ell_2$, $\ell_k=\ell_{k1}+\ell_{k2}+\ell_{k3}$ $(k=1,2)$, $\tilde{\psi}_c(z)=\psi_{c,L}(z+3t)$ and

$$\begin{aligned}
\ell_{11} =&(x_t-2c-3(x_y)^2)\varphi_c' - (c_t-6c_yx_y)\partial_c\varphi_c\,,\quad \ell_{12}=3x_{yy}\varphi_c\,,\\
\ell_{13} =&3c_{yy}\int_z^\infty \partial_c\varphi_c(z_1)dz_1 + 3(c_y)^2\int_z^\infty \partial_c^2\varphi_c(z_1)dz_1\,,\\
\ell_{21} =&(c_t-6c_yx_y)\partial_c\tilde{\psi}_c - (x_t-4-3(x_y)^2)\tilde{\psi}_c'\,,\\
\ell_{22} =&(\partial_z^3-\partial_z)\tilde{\psi}_c - 3\partial_z(\tilde{\psi}_c^2) + 6\partial_z(\varphi_c\tilde{\psi}_c) - 3x_{yy}\tilde{\psi}_c\,,\\
\ell_{23} =&-3c_{yy}\int_z^\infty \partial_c\tilde{\psi}_c(z_1)dz_1 - 3(c_y)^2\int_z^\infty \partial_c^2\tilde{\psi}_c(z_1)dz_1\,,
\end{aligned}$$

$$\begin{aligned}
&N_1=-3v^2\,,\quad N_2=\{x_t-2c-3(x_y)^2\}v+6\tilde{\psi}_cv\,,\\
&N_3=6x_y\partial_yv+3x_{yy}v=6\partial_y(x_yv)-3x_{yy}v\,.
\end{aligned}$$

Here we use the fact that φ_c is a solution of (1.3). We slightly change the definition of $\tilde{\psi}$ from [22] in order to apply the virial identity to $\int_{\mathbb{R}^2}\tilde{\psi}_c(z)v_1^2(t,z,y)\,dzdy$.

Subtracting (4.1) from (4.2), we have

$$\partial_t v_2 = \mathcal{L}_c v_2 + \ell + \partial_z(N_{2,1}+N_{2,2}+N_{2,4}) + N_{2,3}\,, \tag{4.3}$$

where

$$\begin{aligned}
N_{2,1}=-3(2v_1v_2+v_2^2)\,,\quad N_{2,2}=\{x_t-2c-3(x_y)^2\}v_2+6\tilde{\psi}_cv_2\,,\\
N_{2,3}=6\partial_y(x_yv_2)-3x_{yy}v_2\,,\quad N_{2,4}=6(\tilde{\psi}_c-\varphi_c)v_1\,.
\end{aligned}$$

To derive modulation equations on $c(t,y)$ and $x(t,y)$, we differentiate (3.7) with respect to t and substitute (4.3) into the resulting equation. Then for $\eta\in[-\eta_0,\eta_0]$,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} v_2(t,z,y)\overline{g_k^*(z,\eta,c(t,y))}e^{-iy\eta}\,dzdy\\
&=\int_{\mathbb{R}^2}\ell\overline{g_k^*(z,\eta,c(t,y))}e^{-iy\eta}dzdy+\sum_{j=1}^6 II_k^j(t,\eta)=0\,,
\end{aligned} \tag{4.4}$$

where

$$
\begin{aligned}
II_k^1 &= \int_{\mathbb{R}^2} v_2(t,z,y)\mathcal{L}^*_{c(t,y)}(\overline{g_k^*(t,z,c(t,y))e^{iy\eta}})\,dzdy\,,\\
II_k^2 &= -\int_{\mathbb{R}^2} N_{2,1}\overline{\partial_z g_k^*(z,\eta,c(t,y))}e^{-iy\eta}\,dzdy\,,\\
II_k^3 &= \int_{\mathbb{R}^2} N_{2,3}\overline{g_k^*(z,\eta,c(t,y))}e^{-iy\eta}dzdy\\
&\quad + 6\int_{\mathbb{R}^2} v_2(t,z,y)c_y(t,y)x_y(t,y)\overline{\partial_c g_k^*(z,\eta,c(t,y))}e^{-iy\eta}\,dzdy\,,\\
II_k^4 &= \int_{\mathbb{R}^2} v_2(t,z,y)\,(c_t-6c_yx_y)\,(t,y)\overline{\partial_c g_k^*(z,\eta,c(t,y))}e^{-iy\eta}\,dzdy\,,\\
II_k^5 &= -\int_{\mathbb{R}^2} N_{2,2}\overline{\partial_z g_k^*(z,\eta,c(t,y))}e^{-iy\eta}\,dzdy\,,\\
II_k^6 &= -\int_{\mathbb{R}^2} N_{2,4}\overline{\partial_z g_k^*(z,\eta,c(t,y))}e^{-iy\eta}\,dzdy\,.
\end{aligned}
$$

The modulation PDEs of $c(t,y)$ and $x(t,y)$ can be obtained by computing the inverse Fourier transform of (4.4) in η. The leading term of

$$\frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2}\ell_1\overline{g_k^*(z,\eta,c(t,y_1))}e^{i\eta(y-y_1)}\,dzdy_1d\eta$$

is

$$G_k(t,y)=\int_{\mathbb{R}}\ell_1\overline{g_k^*(z,0,c(t,y))}dz\,. \tag{4.5}$$

Since $g_1^*(z,0,c)=\varphi_c(z)$ and $g_2^*(z,0,c)=(c/2)^{3/2}\int_{-\infty}^z\partial_c\varphi_c$, we can compute G_1 and G_2 explicitly.

Lemma 4.1. ([22, Lemma 6.1]) *Let $\mu_1=\frac12-\frac{\pi^2}{12}$ and $\mu_2=\frac{\pi^2}{32}-\frac{3}{16}$. Then*

$$
\begin{aligned}
G_1 =& 16x_{yy}\left(\frac{c}{2}\right)^{3/2}-2(c_t-6c_yx_y)\left(\frac{c}{2}\right)^{1/2}+6c_{yy}-\frac{3}{c}(c_y)^2\,,\\
G_2 =& -2(x_t-2c-3(x_y)^2)\left(\frac{c}{2}\right)^2+6x_{yy}\left(\frac{c}{2}\right)^{3/2}-\frac12(c_t-6c_yx_y)\left(\frac{c}{2}\right)^{1/2}\\
&+\mu_1c_{yy}+\mu_2(c_y)^2\left(\frac{c}{2}\right)^{-1}.
\end{aligned}
$$

We remark that (G_1,G_2) are the dominant part of the modulation equations for c and x. To translate the nonlinear terms $6(c/2)^{1/2}c_yx_y$ and $16x_{yy}\{((c/2)^{3/2}-1\}$ in G_1 into a divergence form, we will make use of the following change of variables. Let $\widetilde{P}_1$ be a projection from $L^2(\mathbb{R})$ onto Y and

$$
\begin{aligned}
& b(t,\cdot)=\frac13\widetilde{P}_1\left\{\sqrt{2}c(t,\cdot)^{3/2}-4\right\}\,,\quad \mathcal{C}_1=\frac12\widetilde{P}_1\left\{c(t,\cdot)^2-4\right\}\widetilde{P}_1\,,\\
& \widetilde{\mathcal{C}}_1=\begin{pmatrix}0&0\\0&\mathcal{C}_1\end{pmatrix},\quad B_1=\begin{pmatrix}2&0\\ \frac12&2\end{pmatrix},\quad B_2=\begin{pmatrix}6&16\\ \mu_1&6\end{pmatrix}.
\end{aligned}
\tag{4.6}
$$

We remark that $b\simeq\tilde{c}=c-2$ if c is close to 2.

Assuming the smallness of the following quantities, we obtain the modulation equations of $c(t,y)$ and $x(t,y)$.

$$\mathbb{M}_{c,x}(T) = \sup_{[0,T]}(\|\tilde{c}(t)\|_Y + \|x_y(t)\|_Y) + \|c_y\|_{L^2(0,T;Y)} + \|x_{yy}\|_{L^2(0,T;Y)}\,,$$
$$\mathbb{M}_1(T) = \sup_{t\in[0,T]}\|v_1(t)\|_{L^2} + \|\mathcal{E}(v_1)^{1/2}\|_{L^2(0,T;W(t))}\,, \quad \mathbb{M}_1'(T) = \sup_{t\in[0,T]}\|\tilde{v}_1(t)\|_{L^3}\,,$$
$$\mathbb{M}_2(T) = \sup_{0\le t\le T}\|v_2(t)\|_X + \|\mathcal{E}(v_2)^{1/2}\|_{L^2(0,T;X)}\,, \quad \mathbb{M}_v(T) = \sup_{t\in[0,T]}\|v(t)\|_{L^2}\,,$$

where $\|v\|_{W(t)} = \|(e^{-\alpha|z|/2} + e^{-\alpha|z+3t+L|})v\|_{L^2(\mathbb{R}^2)}$, L is a large positive constant and

$$\partial_z^{-1}\partial_y v(t,z,y) := \mathcal{F}^{-1}_{\xi,\eta}\left(\frac{\eta}{\xi}\mathcal{F}_{z,y}v(t,\xi,\eta)\right).$$

Proposition 4.2. *There exists a $\delta_3 > 0$ such that if $\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T) + \eta_0 + e^{-\alpha L} < \delta_3$ for a $T \ge 0$, then*

$$\begin{pmatrix} b_t \\ \tilde{x}_t \end{pmatrix} = \mathcal{A}(t,D_y)\begin{pmatrix} b \\ \tilde{x} \end{pmatrix} + \mathcal{N}^1 + O(R_{v_1}) + h.o.t.\,, \tag{4.7}$$

where $\mathcal{A}(t,D_y) = \mathcal{A}_0(D_y) + \mathcal{A}_1(t,D_y)$, $\mu_3 = -\frac{\mu_1}{2} + \frac{3}{4} = \frac{1}{2} + \frac{\pi^2}{24} > 1/8$,

$$\mathcal{A}_0(\eta) = \begin{pmatrix} 2-3\eta^2 & -8\eta^2 \\ 2-\mu_3\eta^2 & -\eta^2 \end{pmatrix} + O(\eta^4)\,, \quad \mathcal{A}_1(t,\eta) = O(e^{-\alpha(3t+L)})\,,$$
$$\mathcal{N}_1 = \widetilde{P}_1\begin{pmatrix} 6(b\tilde{x}_y)_y \\ 2(\tilde{c}-b) + 3(\tilde{x}_y)^2 \end{pmatrix},$$
$$R_{v_1} = \int_{-\eta_0}^{\eta_0}\begin{pmatrix} II_1^6(t,\eta) \\ II_2^6(t,\eta) \end{pmatrix} e^{iy\eta}\,d\eta\,.$$

If $v_2(0) = 0$, then $b(0,\cdot) = 0$ and $x(0,\cdot) = 0$.

The dominant part of $A_0(D_y)$ is determined from (G_1, G_2) in Lemma 4.1. The operator $\mathcal{A}_1$ comes from the auxiliary function $\tilde{\psi}_c$ and it decays exponentially because the interaction between $\tilde{\psi}_c$ and g_k^* becomes exponentially small as $t \to \infty$.

5 À priori estimates for the local speed and the local phase shift

In this section, we will estimate $\mathbb{M}_{c,x}(T)$ assuming the smallness of $\mathbb{M}_{c,x}(T)$, $\mathbb{M}_i(T)$ $(i = 1,2)$, η_0 and $e^{-\alpha L}$.

Lemma 5.1. *There exist positive constants δ_4 and C such that if $\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \eta_0 + e^{-\alpha L} \le \delta_4$, then*

$$\mathbb{M}_{c,x}(T) \le C(\|v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2)\,. \tag{5.1}$$

Outline of the proof of Lemma 5.1. Let us translate (4.7) into a system of b and x_y. By (4.7),

$$\begin{cases} \partial_t\begin{pmatrix} b \\ x_y \end{pmatrix} = A(t,D_y)\begin{pmatrix} b \\ x_y \end{pmatrix} + \text{nonlinear terms}\,, \\ b(0,\cdot) = 0\,, \quad x_y(0,\cdot) = 0\,, \end{cases} \tag{5.2}$$

where $\mu_3 = -\frac{\mu_1}{2} + \frac{3}{4} = \frac{1}{2} + \frac{\pi^2}{24} > 1/8$ and

$$A(t,\eta) = A_*(\eta) + \begin{pmatrix} O(\eta^4) & O(\eta^3) \\ O(\eta^5) & O(\eta^4) \end{pmatrix} + O(e^{-\alpha(3t+L)}), \quad A_*(\eta) = \begin{pmatrix} -3\eta^2 & 8i\eta \\ i\eta(2+\mu_3\eta^2) & -\eta^2 \end{pmatrix}.$$

Let $\omega(\eta) = \sqrt{16 + (8\mu_3 - 1)\eta^2}$, $\lambda_*^{\pm}(\eta) = -2\eta^2 \pm i\eta\omega(\eta)$ and $\Pi_*(\eta) = \frac{1}{4i}\begin{pmatrix} 8i & 8i \\ \eta + i\omega(\eta) & \eta - i\omega(\eta) \end{pmatrix}$. Then

$$\Pi_*(\eta)^{-1} A_*(\eta) \Pi_*(\eta) = \operatorname{diag}(\lambda_*^+(\eta)\,,\, \lambda_*^-(\eta))\,.$$

We remark that $|\omega(\eta) - 4| \lesssim \eta^2$.

By the change of variables $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ and $\begin{pmatrix} b \\ x_y \end{pmatrix} = \Pi_*(D_y)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$,

$$\partial_t \mathbf{b} = \{2\partial_y^2 I + \partial_y \omega(D_y)\sigma_3 + A_2(t, D_y)\}\mathbf{b} + \text{nonlinear terms}\,, \tag{5.3}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_2(t,\eta) = O(\eta^3) + O(e^{-\alpha(3t+L)})$ for $\eta \in [-\eta_0, \eta_0]$.

We decompose the nonlinear terms in (5.3) as

$$\mathcal{N}' + \partial_y(\mathcal{N}^0 + \mathcal{N}'') - \partial_t K(t,y)\,, \tag{5.4}$$

$$\begin{gathered} \mathcal{N}^0 = \widetilde{P}_1 \begin{pmatrix} 4b_1^2 - 4b_1b_2 - 2b_2^2 \\ 2b_1^2 + 4b_1b_2 - 4b_2^2 \end{pmatrix}, \\ \mathcal{N}' \in L^\infty(0,T;Y) \cap L^1(0,T;Y)\,, \quad \mathcal{N}'' \in L^\infty(0,T;Y) \cap L^2(0,T;Y)\,, \\ \sup_{t\in[0,T]} \|K(t,\cdot)\|_Y + \|K\|_{L^2(0,T;Y)} \lesssim \mathbb{M}_1(T)\,, \quad \lim_{t\to\infty} \|K(t,\cdot)\|_Y = 0\,. \end{gathered} \tag{5.5}$$

There is a term of a term $O(R_{v_1})$ related to v_1 in the nonlinearity of (5.3). A crude estimate $\|R_{v_1}\|_{L^2(0,T;Y)} \lesssim \mathbb{M}_1(T)$ is insufficient to obtain a time global monotonicity formula for $\mathbf{b}$ because $\mathbf{b}$ does not necessarily belongs to $L^2(0,T;Y)$. The most problematic part of R_{v_1} is

$$\int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v_1(t,z,y)\varphi_{c(t,y)}^2 e^{-iy\eta}\,dzdyd\eta\,.$$

We use (1.3) and (4.1) to decompose that part into a sum of higher order terms and $O(\partial_t k)$ with

$$k(t,y) = \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v_1(t,z,y_1)\varphi_{c(t,y_1)}(z) e^{i(y-y_1)\eta}\,dzdy_1d\eta\,.$$

Another point is that $\langle \partial_y \mathcal{N}^0, \mathbf{b}\rangle$ may not belong to $L^1(0,T)$ unless v_0 is not strongly localized in the y-direction. In fact, we need to eliminate cubic terms in $\int_{\mathbb{R}} \langle \partial_y \mathcal{N}^0, \mathbf{b}\rangle\,dy$ to obtain a time global bound for $\|\mathbf{b}(t)\|_Y$. For the purpose, we make use of the change of variables

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \mathbf{b} - \left(\frac{1}{2}b_1b_2 + O(k)\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(k)\,. \quad \mathbf{e_1} = \,. \tag{5.6}$$

Taking the $L^2(\mathbb{R})$-inner product between the heat equation of $\mathbf{d}$ and $\mathbf{d}$, we have

$$\sup_{t\in[0,T]} \|\mathbf{d}(t)\|_{L^2}^2 + 4\int_0^T \|\partial_y \mathbf{b}(t)\|_Y^2\,dt \lesssim \|v_0\|_{L^2}^2 + \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T)^2 + \mathbb{M}_2(T)^4\,. \tag{5.7}$$

See [23] for the details. □

6 The $L^2(\mathbb{R}^2)$ estimate

We will estimate $\mathbb{M}_v(T)$ assuming smallness of $\mathbb{M}_{c,x}(T)$, $\mathbb{M}_1(T)$ and $\mathbb{M}_2(T)$.

Lemma 6.1. *Let $\alpha \in (0,1)$ and δ_4 be as in Lemma 5.1. Then there exists a positive constant C such that*

$$\mathbb{M}_v(T) \le C(\|v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)).$$

To prove Lemma 6.1, we will use the following variant of the L^2 conservation law. Let

$$Q(t,v) := \int_{\mathbb{R}^2} \{v(t,z,y)^2 - 2\psi_{c(t,y),L}(z+3t)v(t,z,y)\}\, dzdy.$$

Then for $t \in [0,T]$,

$$\begin{aligned} Q(t,v) = Q(0,v) &+ 2\int_0^t \int_{\mathbb{R}^2} \left(\ell_{11} + \ell_{12} + 6\varphi'_{c(s,y)}(z)\tilde{\psi}_{c(s,y)}(z)\right) v(s,z,y)\, dzdyds \\ &- 6\int_0^t \int_{\mathbb{R}^2} (\partial_z^{-1}\partial_y v)(s,z,y)c_y(s,y)\partial_c\varphi_{c(t,y)}(z)\, dzdy \\ &- 6\int_0^t \int_{\mathbb{R}^2} \varphi'_{c(s,y)}(z)v(s,z,y)^2\, dzdyds - 2\int_0^t \int_{\mathbb{R}^2} \ell\psi_{c(s,y),L}(z+3s)\, dzdyds. \end{aligned}$$

7 Estimates for v_1

In this section, we will give upper bounds of v_1. First, we estimate $\mathbb{M}_1(\infty)$.

Lemma 7.1. *There exist positive constants C and δ_5 such that if $\|v_0\|_{L^2} < \delta_5$, then $\mathbb{M}_1(\infty) \le C\|v_0\|_{L^2}$*

Lemma 7.1 follows from the virial identity and the L^2-conservation law of the KP-II equation. Following the spirit of [18], we use the virial identity to ensure $v_1 \in L^2([0,\infty); L^2_{loc}(\mathbb{R}^2))$. Let $\chi_{+,\varepsilon}(x) = 1 + \tanh \varepsilon x$, $\tilde{x}_1(t)$ be a C^1 function and

$$I_{x_0}(t) = \int_{\mathbb{R}^2} \chi_{+,\varepsilon}(x - \tilde{x}_1(t) - x_0, y)\tilde{v}_1^2(t,x,y)\, dxdy.$$

For any $c_1 > 0$, there exist positive constants ε_0 and δ such that if $\inf_t \tilde{x}_1'(t) \ge c_1$, $\varepsilon \in (0,\varepsilon_0)$ and $\|\tilde{v}_1(0)\|_{L^2} < \delta$, then for any $x_0 \in \mathbb{R}$,

$$I_{x_0}(t) + \nu\int_0^t \int_{\mathbb{R}^2} \chi'_{+,\varepsilon}(x - \tilde{x}_1(s) - x_0)\{(\partial_x\tilde{v}_1)^2 + (\partial_x^{-1}\partial_y\tilde{v}_1)^2 + \tilde{v}_1^2\}(s,x,y)\, dxdyds \le I_{x_0}(0),$$

where $\nu = \frac{1}{2}\min\{3, c_1\}$. Moreover,

$$\lim_{t\to\infty} I_{x_0}(t) = 0 \quad \text{for any } x_0 \in \mathbb{R}. \tag{7.1}$$

See e.g. [25, Lemma 5.3] for a proof.

Next, we estimate $\mathbb{M}_1'(\infty)$.

Lemma 7.2. *There exist positive constants C and δ_5' such that if $\||D_x|^{-1/2}v_0\|_{L^2} + \||D_x|^{1/2}v_0\|_{L^2} + \||D_x|^{-1/2}|D_y|^{1/2}v_0\|_{L^2} < \delta_5'$, then*

$$\mathbb{M}_1'(\infty) \le C(\||D_x|^{-1/2}v_0\|_{L^2} + \||D_x|^{1/2}v_0\|_{L^2} + \||D_x|^{-1/2}|D_y|^{1/2}v_0\|_{L^2}).$$

In order to estimate the L^3-norm of v_1, we apply the small data scattering result for the KP-II equation by [12].

For the sake of self-containedness, let us introduce some notations in [12]. Let $\mathcal{Z}$ be a set of finite partitions $-\infty = t_0 < t_1 < \cdots < t_K = \infty$. We denote by V^p $(1 \le p < \infty)$ the set of all functions $v : \mathbb{R} \to L^2(\mathbb{R}^2)$ such that $\lim_{t\to\pm\infty} v(t)$ exist and for which the norm

$$\|v\|_{V^p} = \left\{ \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2(\mathbb{R}^2)}^p \right\}^{1/p}$$

is finite, where $v(-\infty) := \lim_{t\to-\infty} v(t)$ and $v(\infty) := 0$. We denote by $V^p_{-,rc}$ the closed subspace of every right-continuous function $v \in V^p$ satisfying $\lim_{t\to-\infty} v(t) = 0$. Let $V^p_S := e^{\cdot S}V^p$ and $V^p_{-rc,-,S} := e^{\cdot S}V^p$ with $S = -\partial_x^3 - 3\partial_x^{-1}\partial_y^2$.

Let $\chi \in C_0^\infty(-2,2)$ be an even nonnegative function such that $\chi(\eta) = 1$ for $\eta \in [-1,1]$. Let $\bar{\chi}(t) = \chi(t) - \chi(2t)$ and P_N be a projection defined by $\widehat{P_N u}(\tau,\xi,\eta) = \bar{\chi}(N^{-1}\xi)\hat{u}(\tau,\xi,\eta)$ for $N = 2^n$ and $n \in \mathbb{Z}$. For $s \le 0$, we denote by $\dot{Y}^s$ the closure of $C(\mathbb{R}; H^1(\mathbb{R}^2)) \cap V^2_{-,rc}$ with respect to the norm

$$\|u\|_{\dot{Y}^s} = \left(\sum_N N^{2s} \|P_N u\|_{V_S^2}^2 \right)^{1/2}.$$

We denote by $\dot{Y}^s(0,T)$ the restriction of $\dot{Y}^s$ to the time interval $[0,T]$ with the norm

$$\|u\|_{\dot{Y}^s(0,T)} = \inf\{\|\tilde{u}\|_{\dot{Y}^s} \mid \tilde{u} \in \dot{Y}^s,\ \tilde{u}(t) = u(t) \text{ for } t \in [0,T]\}.$$

Proposition 3.1 and Theorem 3.2 in [12] ensure that higher order Sobolev norms of a solution to (4.1) remain small provided v_0 is small in the higher order Sobolev spaces. Let $T \ge 0$ and

$$I_T(u_1,u_2)(t) = \int_0^t \mathbf{1}_{[0,T]}(s) e^{(t-s)S} \partial_x(u_1u_2)(s)\, ds.$$

Then we have the following.

Lemma 7.3. *Let $s \ge 0$ and u_1, $u_2 \in \dot{Y}^{-1/2}$. Then there exists a positive constant C such that for any $T \in (0,\infty)$,*

$$\||D_x|^s I_T(u_1,u_2)\|_{\dot{Y}^{-1/2}} \le C\||D_x|^s u_1\|_{\dot{Y}^{-1/2}} \|u_2\|_{\dot{Y}^{-1/2}}, \tag{7.2}$$

$$\|\langle D_y\rangle^s I_T(u_1,u_2)\|_{\dot{Y}^{-1/2}} \le C \prod_{j=1,2} \|\langle D_y\rangle^s u_j\|_{\dot{Y}^{-1/2}}. \tag{7.3}$$

Using Lemma 7.3 with $s = 1/2$, we have

$$\sup_{t\ge0} \||D_x|^{1/2}\tilde{v}_1(t)\|_{L^2} \lesssim \||D_x|^{1/2}v_0\|_{L^2},$$

$$\sup_{t\ge0} \||D_x|^{-1/2}|D_y|^{1/2}\tilde{v}_1(t)\|_{L^2} \lesssim \||D_x|^{-1/2}\langle D_y\rangle v_0\|_{L^2},$$

and it follows that

$$\begin{aligned}\|\tilde{v}_1(t)\|_{L^3(\mathbb{R}^2)} &\lesssim \||D_x|^{1/2}\tilde{v}_1(t)\|_{L^2(\mathbb{R}^2)} + \||D_x|^{-1/2}|D_y|^{1/2}\tilde{v}_1(t)\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \||D_x|^{-1/2}\langle D_y\rangle v_0\|_{L^2}.\end{aligned}$$

8 Decay estimates in the exponentially weighted space

In this section, we will estimate $\mathbb{M}_2(T)$ following the line of [22, Chapter 8].

Lemma 8.1. *Let η_0 and α be positive constants satisfying $\nu_0 < \alpha < 2$. Suppose $\mathbb{M}_1'(\infty)$ is sufficiently small. Then there exist positive constants δ_6 and C such that if $\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_v(T) \le \delta_6$,*

$$\mathbb{M}_2(T) \le C(\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)). \tag{8.1}$$

Let $\chi \in C_0^\infty(-2,2)$ be an even nonnegative function such that $\chi(\eta) = 1$ for $\eta \in [-1,1]$. Let $\chi_M(\eta) = \chi(\eta/M)$ and

$$P_{\le M}u := \frac{1}{2\pi}\int_{\mathbb{R}^2} \chi_M(\eta)\hat{u}(\xi,\eta)e^{i(x\xi+y\eta)}\,d\xi d\eta, \quad P_{\ge M} = I - P_{\le M}.$$

To prove Lemma 8.1, we will use linear stability property of line solitons (Proposition 2.1) to the low frequency part $v_<(t) := P_{\le M}v_2(t)$ and make use of a virial type estimate for the high frequency part $v_>(t) := P_{\ge M}v_2(t)$.

8.1 Decay estimates for the low frequency part

Lemma 8.2. *Let η_0 and α be positive constants satisfying $\nu_0 < \alpha < 2$. Suppose that $v_2(t)$ is a solution of (4.3) satisfying $v_2(0) = 0$. Then there exist positive constants δ_6 and C such that if $\mathbb{M}_1(T) + \mathbb{M}_2(T) < \delta_6$ and $M \ge \eta_0$, then*

$$\begin{aligned}&\|P_1(0,2M)v_2\|_{L^\infty(0,T;X)} + \|P_1(0,2M)v_2\|_{L^2(0,T;X)}\\ \le{}& C\left\{\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)(\mathbb{M}_2(T) + \mathbb{M}_v(T))\right\}.\end{aligned} \tag{8.2}$$

Outline of the proof of Lemma 8.2. Let $\tilde{v}_2(t) = P_2(\eta_0, 2M)v_2(t)$. Then

$$\begin{cases} \partial_t\tilde{v}_2 = \mathcal{L}\tilde{v}_2 + P_2(\eta_0,2M)\{\ell + \partial_x(N_{2,1} + N_{2,2} + N_{2,2}' + N_{2,4}) + N_{2,3}\}, \\ \tilde{v}_2(0) = 0, \end{cases} \tag{8.3}$$

where $N_{2,2}' = \{2\tilde{c}(t,y) + 6(\varphi(z) - \varphi_{c(t,y)}(z))\}v_2(t,z,y)$.

Applying Proposition 2.1 to (8.3), we have

$$\begin{aligned}\|\tilde{v}_2(t)\|_X \lesssim{}& \int_0^t e^{-b'(t-s)}(t-s)^{-3/4}\|e^{\alpha z}P_2(\eta_0,2M)N_{2,1}(s)\|_{L^1_zL^2_y}\,ds\\ &+ \int_0^t e^{-b'(t-s)}(t-s)^{-1/2}(\|N_{2,2}(s)\|_X + \|N_{2,2}'(s)\|_X + \|N_{2,4}\|_X)\,ds\\ &+ \int_0^t e^{-b(t-s)}(\|\ell(s)\|_X + \|N_{2,3}(s)\|_X)\,ds.\end{aligned} \tag{8.4}$$

Using the fact that $\|P_{\le M}u\|_{L^1_zL^2_y} \lesssim \sqrt{M}\|u\|_{L^1(\mathbb{R}^2)}$, we have

$$\|e^{\alpha z}P_2(\eta_0,2M)N_{2,1}\|_{L^1_zL^2_y} \lesssim \sqrt{M}(\|v_1\|_{L^2} + \|v_2\|_{L^2})\|v_2\|_X.$$

By the definitions $\mathbb{M}_{c,x}(T)$, $\mathbb{M}_1(T)$ and $\mathbb{M}_2(T)$ and (4.7), we can prove

$$\sup_{t\in[0,T]}\|\tilde{v}_2(t)\|_X + \|\tilde{v}_2(t)\|_{L^2(0,T;X)} \lesssim \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + (\mathbb{M}_v(T) + \mathbb{M}_2(T))\mathbb{M}_2(T).$$

As long as $v_2(t)$ satisfies the orthogonality condition (3.7) and $\tilde{c}(t,y)$ remains small, we have

$$\|\tilde{v}_2(t)\|_X \lesssim \|P_1(0,2M)v_2(t)\|_X \lesssim \|\tilde{v}_2(t)\|_X$$

in exactly the same way as the proof of Lemma 9.2 in [22]. Thus we have (8.2). This completes the proof of Lemma 8.2. □

8.2 Virial estimates for v_2

Next we prove a virial type estimate in the weighted space X in order to estimate the high frequency part of $v_>$. We need the smallness assumption of $\sup_{t\ge 0}\|v_1(t)\|_{L^3(\mathbb{R}^2)}$ to estimate the high frequency part $v_>(t)$.

Lemma 8.3. *Let $\alpha\in(0,2)$ and $v_2(t)$ be a solution to (4.3) satisfying $v_2(0)=0$. Suppose $\mathbb{M}_1'(\infty)$ is sufficiently small. Then there exist positive constants δ_6, M_1 and C such that if $\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)+\mathbb{M}_2(T)+\mathbb{M}_v(T)<\delta_6$ and $M\ge M_1$, then for $t\in[0,T]$,*

$$\|v_2(t)\|_X^2 \le C\int_0^t e^{-M\alpha(t-s)}\left(\|\ell(s)\|_X^2+\|P_{\le M}v_2(s)\|_X^2+\|v_1(s)\|_{W(s)}^2\right)ds\,.$$

The key is for high f To prove virial type identities for v_2, we use the following.

Claim 8.1. *For any $p\in[2,6]$,*

$$\|e^{\alpha x}u\|_{L^p}\le C_1\|u\|_X^{\frac{3}{p}-\frac{1}{2}}(\|\partial_x u\|_X+\|\partial_x^{-1}\partial_y u\|_X+\|u\|_X)^{\frac{3}{2}-\frac{3}{p}}\,. \tag{8.5}$$

Proof of Lemma 8.3. Let $p(z)=e^{2\alpha z}$. Multiplying (4.3) by $2p(z)v_2(t,z,y)$ and integrating the resulting equation by parts, we have for $t\in[0,T]$,

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\mathbb{R}^2}p(z)v_2(t,z,y)^2\,dzdy\right)+\int_{\mathbb{R}^2}p'(z)\left(\mathcal{E}(v_2)-4v_2^3\right)(t,z,y)\,dzdy\\ =&\sum_{k=1}^5 III_k(t)\,.\end{aligned} \tag{8.6}$$

where

$$\begin{aligned}
III_1 =&2\int_{\mathbb{R}^2}p(z)\ell v_2(s,z,y)\,dzdyds\,,\\
III_2 =&-\int_{\mathbb{R}^2}p'(z)\left((\tilde{x}_t(t,y)-3x_y(t,y)^2\right)v_2(t,z,y)^2\,dzdy\,,\\
III_3 =&\int_{\mathbb{R}^2}\left\{p'''(z)+6p(z)^2\left(\frac{\varphi_{c(t,y)}(z)-\psi_{c(t,y),L}(z+3t)}{p(z)}\right)_z\right\}v_2(t,z,y)^2\,dzdy\,,\\
III_4 =&12\int_{\mathbb{R}^2}p'(z)(v_1v_2^2)(t,z,y)\,dzdy+12\int_{\mathbb{R}^2}p(z)(v_1v_2\partial_z v_2)(t,z,y)\,dzdy\,,\\
III_5 =&12\int_{\mathbb{R}^2}\partial_z\left(p(z)v_2(t,z,y)\right)\left(\varphi_{c(t,y)}(z)-\psi_{c(t,y),L}(z+3t)\right)v_1(t,z,y)\,dzdy\,.
\end{aligned}$$

Using Claim 8.1 and the Hölder inequality, we have

$$\left|\int p'(z)v_2(t,z,y)^3dzdy\right|\lesssim\|v_2(t)\|_{L^2}\int_{\mathbb{R}^2}p'(z)\mathcal{E}(v_2(t,z,y))dzdy\,,$$

$$III_4 \lesssim \|v_1(t)\|_{L^3} \int_{\mathbb{R}^2} p'(z) \left((\partial_z v_2)^2 + (\partial_z^{-1}\partial_y v_2)^2 + v_2^2 \right) (t,z,y)\, dzdy\,.$$

For y-high frequencies, the potential term can be absorbed into the left hand side. Indeed, it follows from Plancherel's theorem and the Schwarz inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^2} p'(z) \left((\partial_z v_>)^2 + (\partial_z^{-1}\partial_y v_>)^2 \right) (t,z,y)\, dzdy \\
=&2\alpha \int_{\mathbb{R}^2} \left(|\xi + i\alpha|^2 + \frac{\eta^2}{|\xi+i\alpha|^2} \right) |\mathcal{F}v_>(t,\xi+i\alpha,\eta)|^2\, d\xi d\eta \\
\geq&2M \int_{\mathbb{R}^2} p'(z) v_>(t,z,y)^2\, dzdy\,.
\end{aligned}$$

If $\mathbb{M}_1'(\infty)$ is small enough, we have for $t \in [0,T]$,

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} p(z) v_2(t,z,y)^2\, dzdy + M\alpha \int_{\mathbb{R}} p(z) v_2(t,z,y)^2\, dzdy \\
\leq&\frac{1}{2\alpha} \int_{\mathbb{R}^2} p(z)\ell^2\, dzdy + M\alpha \int_{\mathbb{R}^2} p(z)(v_<)^2(s,z,y)\, dzdy + O\left(\|v_1(t)\|_{W(t)}^2 \right),
\end{aligned} \tag{8.7}$$

provided M is sufficiently large and δ_6 is sufficiently small. Lemma 8.3 follows immediately from (8.7). $\square$

Lemma 8.1 follows from Lemmas 8.2 and 8.3.

9 Proof of Theorem 1.1

If $\mathbb{M}_1'(\infty)$ is sufficiently small, then

$$\begin{aligned}
&\mathbb{M}_1(T) \lesssim \|v_0\|_{L^2}\,, \\
&\mathbb{M}_{c,x}(T) \lesssim \|v_0\|_{L^2} + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2 \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)^2\,, \\
&\mathbb{M}_2(T) \lesssim \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)^2\,, \\
&\mathbb{M}_v(T) \lesssim \|v_0\|_{L^2} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)\,,
\end{aligned}$$

and we have

$$\mathbb{M}_{c,x}(\infty) + \mathbb{M}_1(\infty) + \mathbb{M}_2(\infty) + \mathbb{M}_v(\infty) \lesssim \|v_0\|_{L^2(\mathbb{R}^2)}$$

by a continuation argument.

10 Proof of Theorem 1.2

If $v_0(x,y)$ is polynomially localized, then at $t=0$ we can decompose a perturbed line soliton into a sum of a locally amplified line soliton and a remainder part $v_*(x,y)$ satisfying $\int_{\mathbb{R}} v_*(x,y)\, dx = 0$ for all $y \in \mathbb{R}$.

Lemma 10.1. *Let $c_0 > 0$ and $s > 1$ be constants. There exists a positive constant ε_0 such that if $\varepsilon := \|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)} < \varepsilon_0$, then there exists $c_1(y) \in H^1(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} (\varphi_{c_1(y)}(x) - \varphi_{c_0}(x))\, dx = \int_{\mathbb{R}} v_0(x,y)\, dx\,, \tag{10.1}$$

$$\|c_1(\cdot) - c_0\|_{L^2(\mathbb{R})} \lesssim \left\|\langle x\rangle^{s/2} v_0\right\|_{L^2(\mathbb{R}^2)}\,, \quad \|\partial_y c_1(\cdot)\|_{H^1(\mathbb{R})} \lesssim \left\|\langle x\rangle^{s/2} v_0\right\|_{H^1(\mathbb{R}^2)}\,, \tag{10.2}$$

$$\|v_*\|_{L^2(\mathbb{R}^2)} \lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}\,, \quad \|\partial_x^{-1} v_*\|_{L^2} + \|v_*\|_{H^1(\mathbb{R}^2)} \lesssim \|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)}\,, \tag{10.3}$$

where $v_*(x,y) = v_0(x,y) + \varphi_{c_0}(x) - \varphi_{c_1(y)}(x)$.

To prove Theorem 1.2, we modify the definitions of $v_1(t,z,y)$, $v_2(t,z,y)$ $c(t,y)$ and $x(t,y)$ as follows. Let $\tilde{v}_1$ be a solution of (1.1) satisfying $\tilde{v}_1(0,x,y) = v_*(0,x,y)$. Lemmas 7.1 and 10.1 imply

$$\mathbb{M}_1(\infty) \lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}\,.$$

Lemma 7.2 and (10.3) imply

$$\mathbb{M}_1'(\infty) \lesssim \|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)}\,.$$

Let $\tilde{u}(t,x,y) = u(t,x,y) - \tilde{v}_1(t,x,y)$. Then $\tilde{u}(0,x,y) = \varphi_{c_1(y)}(x)$. By Lemma 10.1,

$$\|u(0,x,y) - \varphi_{c_0}(x)\|_X \lesssim \|c_1(\cdot) - c_0\|_{L^2(\mathbb{R})} \lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}\,,$$

and by the implicit function theorem, there exist a $T > 0$ and $(v_2(t), \tilde{c}(t), \tilde{x}(t)) \in X \times Y \times Y$ satisfying (3.1), (3.4) and (3.7) for $t \in [0,T]$, where $\tilde{c}(t,y) = c(t,y) - c_0$ and $\tilde{x}(t,y) = x(t,y) - 2c_0 t$. Clearly, we have

$$\|v_2(0)\|_{X\cap L^2(\mathbb{R}^2)} + \|\tilde{c}(0)\|_Y \lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}\,, \quad x(0,\cdot) = 0\,,$$

and following the proof of Lemmas 5.1, 6.1 and 8.1, we can prove

$$\begin{gathered}
\mathbb{M}_{c,x}(T) \lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2\,, \\
\mathbb{M}_v(T) \lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)\,, \\
\mathbb{M}_2(T) \lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)\,.
\end{gathered}$$

Thus we can prove Theorem 1.2 in exactly the same way as Theorem 1.1.

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